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Thermoelastic interactions without energy dissipation in a semi-infinite medium: Analytical-numerical solution

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Abstract — **The present work is aimed at the study of thermoelastic interactions in a semi-infinite medium in the context of the Green and Naghdi theory of type II. The governing equations are expressed in Laplace transform domain and solved in the domain by analytical method and finite element method. The solutions of the problem in the physical domain are obtained by using a numerical method for the inversion of the Laplace transforms based on Stehfest's method. Numerical results for the temperature distribution, displacement and thermal stress are represented graphically.**

Keywords: *Laplace transform, Finite element method, Green and Naghdi theory.*

I. INTRODUCTION

For the last three decades, the generalized theories of thermoelasticity, which admit the finite speed of thermal signal, have been the center of interest of active research. Cylindrical shell structure is a common structure type that can be used in applications involving aerospace, submarine structures, nuclear reactors as well as chemical pipes. When the structures are exposed to a temperature field, the thermal stresses are then induced. The research for thermoelastic problems, especially for dynamic Thermoelastic problems, is of increasing interest in engineering science and many works have been done. The generalized theories of thermoelasticity remove the paradox of infinite speed of heat propagation inherent in the conventional coupled dynamical theory of thermoelasticity introduced by Biot [1]. The theory of couple thermoelasticity was extended by Lord and Shulman [2] and Green and Lindsay [3] by including the thermal relaxation time in constitutive relations. The theory was extended for anisotropic body by Dhaliwal and Sherief [4]. During the last three decades a number of investigations [5-8] have been carried out using the aforesaid theories of generalized thermoelasticity. Note that in most of the earlier studies, mechanical or thermal loading on the bounding surface was considered to be in the form of a shock. Youssef [9-10] has formulated a problem of an infinite body having a cylindrical cavity using the Laplace transform technique and the same has been solved numerically based on the Fourier series expansion. The exact solution of the generalized thermoelasticity theory governing equations for a coupled and non-linear/linear exists only for very special and simple initial and boundary problem. In view of calculating general problems, a numerical solution technique is to be used. For this reason the finite element method is chosen. The method

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of weighted residuals offers us the formulation of the finite element equations and we obtain a best approximated solutions to linear and nonlinear ordinary and partial differential equations. Applying this method basically involves three steps. The first step is to assume the general behavior of the unknown field variables in such a way as satisfy the given differential equations. Substitution of these approximating functions into the differential equations and boundary conditions result in some errors, called the residual. This residual has to vanish in an average sense over the solution domain. The second step is the time integration. The time derivatives of the unknown variables have to be determined by former results. The third step is to solve the equations resulting from the first and the second step by the solving algorithm of the finite element program. Abbas and Abbas et. al [11-17] applied the finite element method in different problems.

The present investigation is devoted to study the thermoelastic interactions in a semi-infinite medium without energy dissipation by analytical method (Exact solution) and numerical method (finite element method). Numerical results for the temperature distribution, displacement and thermal stress are represented graphically. Finally, the accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

II. BASIC EQUATION AND FORMULATION

In the context of the Green and Naghdi-theory of type II, the field equations for linear equations in homogenous and isotropic thermoelastic continuum, the generalized field equations can be presented in a unified form as [10]

$$\rho\frac{\partial^2 u_i}{\partial t^2}=(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}-\gamma T_{,i}+F_i. \tag{1}$$

$$K_{ij}^{*}T_{,ii}=\frac{\partial^2}{\partial t^2}\left(\rho c_e T+\gamma T_0 u_{j,j}\right). \tag{2}$$

The constitutive equations are given by

$$\tau_{ij}=\lambda u_{i,j}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)-\gamma\left(T-T_0\right)\delta_{ij}. \tag{3}$$

Where λ and μ are Lame's constants, ρ is the density of medium, c_e is specific heat at constant strain, t is the time, T is the temperature, T_0 is the reference temperature, $\mathrm{K}_{\mathrm{ij}}^{*}$ are the material constant characteristic of the theory, δ_{ij} is the kronecker symbol, τ_{ij} are the components of stress tensor, u_i are the components of displacement vector, F_i are the body force vector, $\gamma=(3\lambda+2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion. It assumed that the state of the medium depends only on x and the time variable t. It is assumed that there are no body forces and heat sources in the medium and that the plane $x=0$ is taken to be traction free. Thus the field equations (1)-(3) in a one-dimensional case can be put as

$$\left(\lambda+2\mu\right)\frac{\partial^2 u}{\partial x^2}-\gamma\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

$$\frac{\partial^2 T}{\partial x^2}=\frac{\partial^2}{\partial t^2}\left(\frac{\rho c_e}{K^*}T+\frac{\gamma T_0}{K^*}\frac{\partial u}{\partial x}\right), \tag{5}$$

$$\tau_{xx}=\left(\lambda+2\mu\right)\frac{\partial u}{\partial x}-\gamma\left(T-T_0\right). \tag{6}$$

For convenience, we shall use the following non-dimensional variables:

$$(x^\circ,u^\circ)=\frac{1}{b}(x,u),\ T^\circ=\frac{T-T}{T_0},t^\circ=\frac{c}{b}t,\tau_{xx}^\circ=\frac{\tau_{xx}}{\lambda+2\mu}.$$

where $c^2=\dfrac{\lambda+2\mu}{\rho}$. Into equations (4)-(6), one may obtain (after dropping the superscript ° for convenience)

$$\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial T}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{7}$$

$$\frac{\partial^2 T}{\partial x^2}=\frac{\partial^2}{\partial t^2}\left(\varepsilon_1 T+\varepsilon_2\frac{\partial u}{\partial x}\right), \tag{8}$$

$$\tau_{xx}=\frac{\partial u}{\partial x}-\beta T, \tag{9}$$

where $\beta=\frac{T_0\gamma}{\rho c^2}, \varepsilon_1=\frac{\rho c_e c^2}{k^*}, \varepsilon_2=\frac{c^2\gamma}{k^*}$.

The non-dimensional forms of the initial and boundary condition are:

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=0, T(x,0)=\frac{\partial T(x,0)}{\partial t}=0, \tag{10}$$

$$\tau_{xx}(0,t)=0,\ T(0,t)=T_1H(t), \tag{11}$$

where $H(t)$ denotes the Heaviside unit step function.

III. BASIC EQUATIONS IN THE LAPLACE TRANSFORM DOMAIN

Applying the Laplace transform for equations (7)-(9) define by the formula

$$\bar{f}(s)=L[f(t)]=\int_0^\infty f(t)e^{-st}dt. \tag{12}$$

Hence, we obtain the following system of differential equations

$$\frac{d^2\bar{u}}{dx^2}-f_2\frac{d\bar{T}}{dx}-f_1\bar{u}=0, \tag{13}$$

$$\frac{d^2\bar{T}}{dx^2}-f_3\bar{T}-f_4\frac{d\bar{u}}{dx}=0, \tag{14}$$

$$\bar{\tau}_{xx}=\frac{d\bar{u}}{dx}-\beta\bar{T}, \tag{15}$$

with $f_1=s^2, f_2=\beta, f_3=s^2\varepsilon_1\ and\, f_4=s^2\varepsilon_2$, and s denotes the Laplace transform parameter. All the state functions initially are equal to zero, and the boundary conditions (11) become

$$\bar{\tau}_{xx}(0,s)=0,\ \bar{T}(0,s)=\frac{T_1}{s}. \tag{16}$$

IV. EXACT SOLUTION

Eliminating $\bar{T}$ from the equations (13) and (14) we get

$$\left(\frac{d^4}{dx^4}-(f_1+f_3+f_2f_4)\frac{d^2}{dx^2}+f_1f_3\right)\bar{u}=0, \tag{17}$$

The solutions of Equation (17) bounded at infinity can be written in the form:

$$\bar{u}=A_1e^{-m_1x}+A_2e^{-m_2x}, \tag{18}$$

where A_1 and A_2 are parameters depending on s to be determined from the boundary conditions, m_1 and m_2 are the roots with positive real parts of the characteristic equation

$$m^4-(f_1+f_3+f_2f_4)m^2+f_1f_3=0, \tag{19}$$

m_1 and m_2 are given by

$$m_1=\sqrt{\frac{1}{2}\left[f_1+f_3+f_2f_4+\sqrt{(f_1+f_3+f_2f_4)^2-4f_1f_3}\right]},$$

$$m_2=\sqrt{\frac{1}{2}\left[f_1+f_3+f_2f_4-\sqrt{(f_1+f_3+f_2f_4)^2-4f_1f_3}\right]},$$

From equation (18) into equations (13) and (14), the expression for temperature can be written in the form

$$\bar{T}=B_1e^{-m_1x}+B_2e^{-m_2x}, \tag{20}$$

where $B_i=\eta_iA_i,\ \eta_i=\frac{f_1+f_2f_4-m_i^2}{f_2f_3}m_i,\quad i=1,2.$

Substituting from equations (19) and (20) into equation (15), we obtain

$$\bar{\tau}_{xx}=-(m_1+\beta\eta_1)A_1e^{-m_1x}-(m_2+\beta\eta_2)A_2e^{-m_2x}. \tag{21}$$

From the boundary conditions (16), it follows that

$$A_1=\frac{T_1(m_2+\beta\eta_2)}{s\left[(m_2+\beta\eta_2)\eta_1-(m_1+\beta\mu_1)\eta_2\right]},$$

$$A_2=-\frac{(m_1+\beta\eta_1)}{(m_2+\beta\eta_2)}A_1.$$

V. NUMERICAL SOLUTION

In order to investigate the thermoelastic interactions in a semi-infinite medium without energy dissipation, the finite element method (FEM) [18] is adopted due to its flexibility in modeling layered structures and its capability in obtaining full field numerical solution. The governing equations (13) and (14) are coupled with boundary conditions (16). The numerical values of the dependent variables like displacement $\bar{u}$ and the temperature $\bar{T}$ are obtained at the interesting points which are called degrees of freedom. The weak formulations of the non-dimensional governing equations are derived. The set of independent test functions to consist of the displacement $\delta\bar{u}$ and the temperature $\delta\bar{T}$ is prescribed. The governing equations are multiplied by independent weighting functions and then are integrated over the spatial domain with the boundary. Applying integration by parts and making use of the divergence theorem reduce the order of the spatial derivatives and allows for the application of the boundary conditions. The same shape functions are defined piecewise on the elements. Three nodes of quadrilateral elements are used. The shape function is usually denoted by the letter N and is usually the coefficient that appears in the interpolation polynomial. A shape function is written for each individual node of a finite element and has the property that its magnitude is 1 at that node and 0 for all other nodes in that element. We assume that the master element has its local coordinates in the range $[-1,1]$. In our case, the one-dimensional quadratic elements are used, which given by: Linear shape functions

$$N_1=\frac{1}{2}(1-\xi), \quad N_2=\frac{1}{2}(1+\xi),$$

Quadratic shape functions

$$N_1=\frac{1}{2}(\xi^2-\xi), \quad N_2=1-\xi^2, \quad N_3=\frac{1}{2}(\xi^2+\xi),$$

VI. NUMERICAL INVERSION OF THE LAPLACE TRANSFORMS

For the final solution of temperature, displacement and stress distributions in the time domain, we adopt a numerical inversion method based on the Stehfest [19]. In this method, the inverse $f(t)$ of the Laplace transform $f(s)$ is approximated by the relation

$$f(t)=\frac{ln\,2}{t}\sum_{i=1}^{n}V_i F\left(\frac{ln\,2}{t}i\right), \tag{26}$$

Where V_i is given by the following equation:

$$V_i=(-1)^{\left(\frac{n}{2}+1\right)}\sum_{k=\frac{i+1}{2}}^{min\left(i,\frac{n}{2}\right)}\frac{k^{\left(\frac{n}{2}+1\right)}(2k)!}{\left(\frac{n}{2}-k\right)!\,k!\,(i-k)!\,(2k-1)!}. \tag{27}$$

The parameter n is the number of terms used in the summation in equation (26) and should be optimized by trial and error. Increasing n increases the accuracy of the result up to a point, and then the accuracy declines because of increasing round-off errors. An optimal choice of $10\le n\le 14$ has been reported by Lee et al. for some problem of their interest [20].

VII. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the problem, the copper material was chosen for purposes of numerical evaluations. The physical data which given as [10]

$$\lambda=7.76\times10^{10}(kg)(m)^{-1}(s)^{-2},$$
$$\mu=3.86\times10^{10}(kg)(m)^{-1}(s)^{-2},$$

$$K=3.68\times10^{2}(kg)(m)(K)^{-1}(s)^{-3},$$
$$c_e=3.831\times10^{2}(m)^{2}(K)^{-1}(s)^{-2},$$
$$T_0=293(K), T_1=1,$$

$$\rho=8.954x10^{3}(kg)(m)^{-3},$$
$$\alpha_t=17.8\times10^{-6}(K)^{-1}.$$

Ibrahim A. Abbas JOURNAL OF PHYSICS VOL. 2 NO. 1 Febraury (2013) PP. 4 - 9

The results for displacement, temperature and stress has been carried out by taking $T_1 = 1$. Figures 1, 2 and 3 exhibit the variation of the displacement, temperature and stress with space x for different values of time ($t = 0.1, 0.2, 0.3, 0.4$). It is obvious from figure 1 that the displacement is negative at $x = 0$ where its magnitude is maximum. The displacement increases from the negative value to a positive value. In the positive values, the displacement has a peak value that depends on the values of the time. It is obvious from figure 2 that the temperature decreases with the increase of the space but they increase when increasing the time. It is obvious from figure 3 which gives the stress variation at different instants of time with the space. Its magnitude increases from zero to a maximum value after that decreases rapidly as x increases.

Finally, figures 1-3 illustrates the solution obtained numerically by finite element method (—o—) overlaid onto the solution obtained analytically (——). The accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

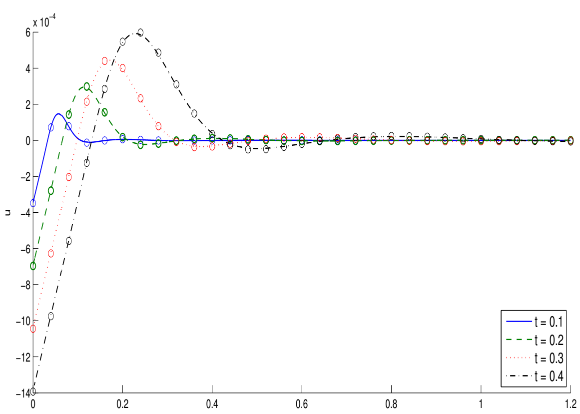

Figure 1. Variation of displacement

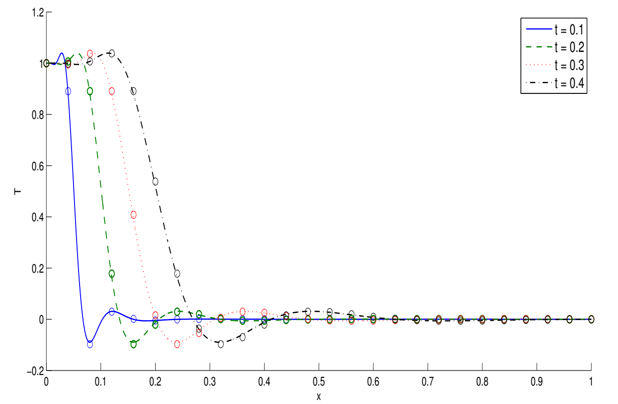

Figure 2. Variation of temperature

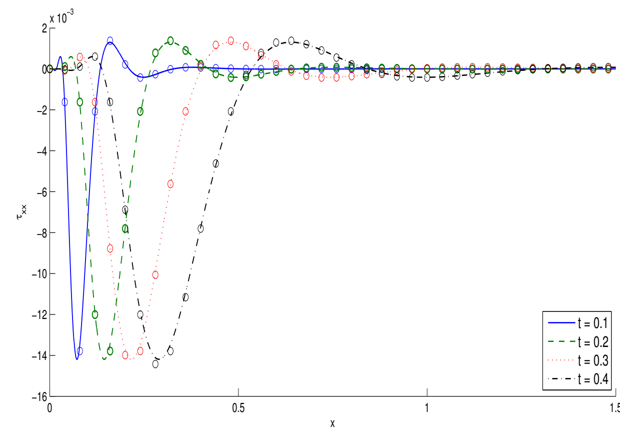

Figure 3. Variation of stress

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